

$\mathcal{N} = 1$ Supersymmetric Double Field Theory

Olaf Hohm¹ and Seung Ki Kwak²

¹*Arnold Sommerfeld Center for Theoretical Physics*

Theresienstrasse 37

D-80333 Munich, Germany

olaf.hohm@physik.uni-muenchen.de

²*Center for Theoretical Physics*

Massachusetts Institute of Technology

Cambridge, MA 02139, USA

sk_kwak@mit.edu

Abstract

We construct the $\mathcal{N} = 1$ supersymmetric extension of double field theory for $D = 10$, including the coupling to an arbitrary number n of abelian vector multiplets. This theory features a local $O(1, 9 + n) \times O(1, 9)$ tangent space symmetry under which the fermions transform. It is shown that the supersymmetry transformations close into the generalized diffeomorphisms of double field theory.

1 Introduction

Double field theory is an approach to make the T-duality group $O(D, D)$ a manifest symmetry of the massless sector of string theory by doubling the D space-time coordinates [1–4]. (See [5–23] for earlier work and further developments.) Thus, for $D = 10$ the theory features a global $O(10, 10)$ symmetry and depends formally on 20 coordinates, but consistency requires an $O(10, 10)$ invariant constraint that locally removes the dependence on half of the coordinates. Here we will construct the $\mathcal{N} = 1$ supersymmetric extension of double field theory for $D = 10$.

Naively, one may suspect that such a construction is impossible, for there simply are no supersymmetric theories beyond eleven dimensions. The aforementioned constraint, however, makes the supersymmetric extension feasible, because for every solution of the constraint, locally the fields depend only on ten coordinates.

The formulation of double field theory that is most useful for our present purpose is the frame or vielbein formulation. The double field theory can be written in terms of the generalized metric

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix}, \quad (1.1)$$

that takes values in $O(10, 10)$ and combines the space-time metric g_{ij} and the Kalb-Ramond 2-form b_{ij} . As usual, we may introduce frame fields E_M^A and write

$$\mathcal{H}_{MN} = E_M^A E_N^B \hat{\eta}_{AB}, \quad \hat{\eta}_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix}, \quad (1.2)$$

where η denotes the standard Minkowski metric, and we have split the flat or frame indices as $A = (a, \bar{a})$. Consequently, in the frame formulation there is an $O(1, 9)_L \times O(1, 9)_R$ ‘tangent space’ gauge symmetry, with $a, b \dots = 0, \dots, 9$ and $\bar{a}, \bar{b} \dots = 0, \dots, 9$ denoting $O(1, 9)_L$ and $O(1, 9)_R$ vector indices, respectively. Such a frame formalism has been developed by Siegel prior to the generalized metric formulation [5]. Actually, Siegel’s formalism allows also for the larger tangent space group $GL(D) \times GL(D)$, but here we will restrict to the Lorentz subgroups in order to be able to define the corresponding spinor representations. In this formalism one may introduce connections for the local frame symmetry and construct invariant curvatures. This, in turn, allows one to write an Einstein-Hilbert like action based on a generalized curvature scalar \mathcal{R} , which provides an equivalent definition of double field theory,

$$S = \int d^{10}x d^{10}\tilde{x} e^{-2d} \mathcal{R}(E, d), \quad (1.3)$$

where we defined $e^{-2d} = \sqrt{g}e^{-2\phi}$. In the frame formulation the theory has a global $O(10, 10)$ symmetry, a $O(1, 9)_L \times O(1, 9)_R$ gauge invariance and a ‘generalized diffeomorphism’ symmetry.

In this paper we will introduce fermions that, as usual in supergravity, are scalars under (generalized) diffeomorphisms and $O(10, 10)$, but which transform under the local tangent space group $O(1, 9)_L \times O(1, 9)_R$. The fermionic sector of supergravity is thereby rewritten in a way that enlarges the local Lorentz group. Similar attempts have in fact a long history, going back to the work of de Wit and Nicolai in the mid 80’s, in which they showed that 11-dimensional supergravity can be reformulated such that it permits an enhanced tangent space symmetry [24].

More recently, a very interesting paper appeared which showed in the context of generalized geometry that type II supergravity can be reformulated such that it permits a doubled Lorentz group [25], as in double field theory, and our results are closely related (see also [18]).

We will introduce a gravitino field Ψ_a that is a spinor under $O(1,9)_R$ and a vector under $O(1,9)_L$, together with a dilatino ρ , that is a spinor under $O(1,9)_R$. The minimally supersymmetric extension of (1.3) can then be written as

$$S_{\mathcal{N}=1} = \int d^{10}x d^{10}\tilde{x} e^{-2d} \left(\mathcal{R}(E, d) - \bar{\Psi}^a \gamma^{\bar{b}} \nabla_{\bar{b}} \Psi_a + \bar{\rho} \gamma^{\bar{a}} \nabla_{\bar{a}} \rho + 2 \bar{\Psi}^a \nabla_a \rho \right). \quad (1.4)$$

Here, the $\gamma^{\bar{a}}$ are ten-dimensional gamma matrices, which have to be thought of as gamma matrices of $O(1,9)_R$, so that all suppressed spinor indices in (1.3) are $O(1,9)_R$ spinor indices. Moreover, the covariant derivatives ∇ are with respect to the connections introduced by Siegel [5], and therefore the action is manifestly $O(1,9)_L \times O(1,9)_R$ invariant.

We will show that (1.4), up to field redefinitions, reduces precisely to the standard minimal $\mathcal{N} = 1$ action in ten dimensions. In this paper we will not consider higher-order fermi terms. Formally, (1.4) is contained in the results of [25] through the straightforward truncation from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. The main difference between generalized geometry, which was the starting point in [25], and double field theory is that in the former the coordinates are not doubled but only the tangent space. Consequently, in generalized geometry only the tangent space symmetry is enhanced, while double field theory features also a global $O(D, D)$ symmetry. With the fermions being singlets under $O(D, D)$, this symmetry is somewhat trivially realized on the fermionic sector, and therefore our results for the minimal $\mathcal{N} = 1$ theory are largely contained in those of generalized geometry given in [25]. In the spirit of double field theory, however, it is reassuring to verify closure of the supersymmetry transformations into generalized diffeomorphisms and supersymmetric invariance of (1.4), both modulo the $O(D, D)$ invariant constraint. This will be done in sec. 2 of this paper.

As the main new result, we will present in sec. 3 the double field theory extension of $\mathcal{N} = 1$ supergravity in $D = 10$ coupled to an arbitrary number n of (abelian) vector multiplets. For $n = 16$ this is the low-energy effective action of heterotic superstring theory truncated to the Cartan subalgebra of $SO(32)$ or $E_8 \times E_8$. As has been shown in [9], the coupling of gauge vectors A_i^α can be neatly described by enlarging the generalized metric (1.1) to an $O(10+n, 10)$ matrix that naturally contains the A_i^α . In the frame formulation this theory features, in addition, a $O(1, 9+n) \times O(1, 9)$ tangent space symmetry. The fermionic fields will still be spinors under $O(1, 9)$, but Ψ_a is now a vector under $O(1, 9+n)$. Remarkably, it turns out that the same action (1.4), but written with respect to these enlarged fields, reproduces precisely the $\mathcal{N} = 1$ supergravity coupled to abelian vector multiplets, with the gauginos originating from the additional components of the Ψ_a .

Let us finally mention that in the work of Siegel the construction proceeds immediately in $\mathcal{N} = 1$ superspace [5]. Therefore, our results on the $\mathcal{N} = 1$ theory, including the coupling to vector multiplets, must be related to the construction of Siegel, but we have not been daring enough to attempt an explicit verification.

Note added: After the submission of the first version of this paper to the arxiv, [26] appeared, which overlaps with our section 2.

2 Minimal $\mathcal{N} = 1$ Double Field Theory for $D = 10$

In this section we introduce the minimal $\mathcal{N} = 1$ theory. First, we review the vielbein formalism with local $O(1,9)_L \times O(1,9)_R$ symmetry. Second, we introduce the $\mathcal{N} = 1$ double field theory and prove its supersymmetric invariance. In the third subsection we verify that it reduces to conventional $\mathcal{N} = 1$ supergravity upon setting the new derivatives to zero.

2.1 Vielbein formulation with local $O(1,9) \times O(1,9)$ symmetry

We start by reviewing some generalities on the vielbein formulation of double field theory, which is contained in Siegel's frame formalism [5]. We refer to [7] for a self-contained presentation of this formulation. The fundamental bosonic fields are the frame field E_A^M and the dilaton d that depend both on doubled coordinates $X^M = (\tilde{x}_i, x^i)$. The frame field is subject to local $O(1,9)_L \times O(1,9)_R$ transformations acting on the index $A = (a, \bar{a})$ and global $O(10,10)$ transformations acting on the index M , which read infinitesimally

$$\delta E_A^M = k^M_N E_A^N + \Lambda_A^B(X) E_B^M, \quad k \in \mathfrak{o}(10,10), \quad \Lambda(X) \in \mathfrak{o}(1,9)_L \oplus \mathfrak{o}(1,9)_R, \quad (2.1)$$

where the parameters take values in the respective Lie algebras. The double field theory is invariant under a ‘generalized diffeomorphism’ symmetry parameterized by $\xi^M = (\tilde{\xi}_i, \xi^i)$ that combines the b -field 1-form gauge parameter $\tilde{\xi}_i$ with the vector-valued diffeomorphism parameter ξ^i ,

$$\delta_\xi E_A^M = \hat{\mathcal{L}}_\xi E_A^M \equiv \xi^N \partial_N E_A^M + (\partial^M \xi_N - \partial_N \xi^M) E_A^N. \quad (2.2)$$

Here, $\partial_M = (\tilde{\partial}^i, \partial_i)$ are the doubled partial derivatives. The right-hand side of (2.2) defines a generalized Lie derivative that can similarly be defined for an $O(D,D)$ tensor with an arbitrary number of upper and lower indices. On the dilaton d these gauge transformations read

$$\delta_\xi d = \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M. \quad (2.3)$$

The gauge transformations close and leave the action invariant modulo the ‘strong constraint’

$$\eta^{MN} \partial_M \partial_N = 0, \quad \eta^{MN} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad (2.4)$$

when acting on arbitrary fields and parameters and all their products. Here, η_{MN} denotes the $O(10,10)$ invariant metric, which will be used to raise and lower $O(10,10)$ indices. This constraint implies that locally all fields depend only on half of the coordinates, for instance only on the x^i .

We have to impose covariant constraints on the frame field in order to describe only the physical degrees of freedom. These constraints are written in terms of the tangent space metric

$$\mathcal{G}_{AB} \equiv E_A^M E_B^N \eta_{MN}, \quad (2.5)$$

resulting from the $O(10,10)$ invariant metric η , and which will be used to raise and lower flat indices. We require the $O(1,9)_L \times O(1,9)_R$ covariant constraints

$$\mathcal{G}_{a\bar{b}} = 0, \quad \mathcal{G}_{ab} = \eta_{ab}, \quad \mathcal{G}_{\bar{a}\bar{b}} = -\eta_{\bar{a}\bar{b}}. \quad (2.6)$$

Note that the relative minus sign entering here is necessary due to the (10,10) signature of \mathcal{G}_{AB} . It is a matter of convention to which metric we assign the minus sign, but once the choice is made the symmetry between unbarred and barred indices is broken. Since flat indices are raised and lowered with \mathcal{G}_{AB} , (2.6) leads to some unconventional signs when comparing below to standard expressions for, say, the spin connection. We will comment on this in due course.

A particular solution of these constraints, giving rise to the generalized metric (1.1) according to (1.2), is given by

$$E_A{}^M = \begin{pmatrix} E_{ai} & E_a{}^i \\ E_{\bar{a}i} & E_{\bar{a}}{}^i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_{ia} + b_{ij}e_a{}^j & e_a{}^i \\ -e_{i\bar{a}} + b_{ij}e_{\bar{a}}{}^j & e_{\bar{a}}{}^i \end{pmatrix}, \quad (2.7)$$

where e is the vielbein of the conventional metric, $g = e \eta e^T$. We stress that when writing (2.7) the tangent space symmetry is gauge-fixed to the diagonal subgroup of $O(1,9)_L \times O(1,9)_R$, as is clear from the fact that e carries in (2.7) both unbarred and barred indices. In order to define the supersymmetric double field theory, however, (2.7) is never used. Rather, we view the (constrained) vielbein $E_A{}^M$ as the fundamental field and so the construction is manifestly invariant under two copies of the local Lorentz group. It is only when comparing to the standard formulation of supergravity that we have to use (2.7) and to partially gauge-fix.¹

Let us now turn to the definition of connections and covariant derivatives. We first note that the partial derivative of a field S that transforms as a scalar under ξ^M , i.e.,

$$\delta_\xi S = \xi^M \partial_M S, \quad (2.8)$$

transforms covariantly with a generalized Lie derivative [7]. This does not hold for higher tensors, which in turn necessitates the introduction of covariant derivatives. Given the frame field $E_A{}^M$, we introduce the ‘flattened’ partial derivative²

$$E_A \equiv \sqrt{2} E_A{}^M \partial_M. \quad (2.9)$$

We can then introduce $O(1,9)_L \times O(1,9)_R$ covariant derivatives

$$\nabla_A V_B = E_A V_B + \omega_{AB}{}^C V_C, \quad \nabla_A V^B = E_A V^B - \omega_{AC}{}^B V^C, \quad (2.10)$$

where we stress that the only non-trivial connections are $\omega_{Ab}{}^c$ and $\omega_{\bar{A}\bar{b}}{}^{\bar{c}}$.

Next, we briefly summarize which connection components can be determined in terms of $E_A{}^M$ and d upon imposing covariant constraints. First, in order to be compatible with the constancy of the tangent space metric \mathcal{G}_{AB} , the symmetric part $\omega_{A(BC)}$, where indices have been lowered with \mathcal{G} , is zero. Thus, ω_{ABC} is antisymmetric in its last two indices. Second, we can impose a generalized torsion constraint, which reads

$$\mathcal{T}_{ABC} \equiv \Omega_{ABC} + 3\omega_{[ABC]} = 0, \quad (2.11)$$

where we introduced the ‘generalized coefficients of anholonomy’

$$\Omega_{ABC} = 3f_{[ABC]}, \quad f_{ABC} \equiv (E_A E_B{}^M) E_{CM}. \quad (2.12)$$

¹This differs from the construction in [25] and [16, 18], where two independent vielbein fields are introduced, one transforming under $O(1,9)_L$ and one transforming under $O(1,9)_R$.

²Here we introduced a factor of $\sqrt{2}$ for later convenience. With the constraints on the connections to be imposed below, the covariant derivatives ∇_A given here are $\sqrt{2}$ times the covariant derivatives in [7].

We note that f_{ABC} is antisymmetric in its last two indices as a consequence of the constancy of \mathcal{G}_{AB} . Specializing the constraint (2.11) to $\mathcal{T}_{a\bar{b}\bar{c}} = 0$ and $\mathcal{T}_{\bar{a}bc} = 0$, we derive the following solution for the ‘off-diagonal’ components

$$\omega_{a\bar{b}\bar{c}} = -\Omega_{a\bar{b}\bar{c}}, \quad \omega_{\bar{a}bc} = -\Omega_{\bar{a}bc}. \quad (2.13)$$

For later use let us determine these connection components for the gauge choice (2.7) of the frame field, setting $\tilde{\partial}^i = 0$. We compute with (2.12)

$$f_{a\bar{b}\bar{c}} = e_a^i e_{[\bar{b}}^j \partial_i e_{j\bar{c}]} + \frac{1}{2} e_a^i e_{\bar{b}}^k e_{\bar{c}}^j \partial_i b_{jk}, \quad f_{\bar{a}bc} = e_{\bar{b}}^i e_{(a}^j \partial_i e_{j\bar{c})} + \frac{1}{2} e_{\bar{b}}^i e_a^k e_{\bar{c}}^j \partial_i b_{jk}, \quad (2.14)$$

from which we derive

$$\omega_{a\bar{b}\bar{c}} = -\omega_{a\bar{b}\bar{c}}^L(e) + \frac{1}{2} e_a^i e_{\bar{b}}^j e_{\bar{c}}^k H_{ijk}, \quad (2.15)$$

where ω^L denotes the standard Levi-Civita spin connection expressed in terms of the vielbein,

$$\omega_{a\bar{b}\bar{c}}^L(e) = e_{[a}^i e_{\bar{b}}^j \partial_i e_{j\bar{c}]} - e_{[\bar{b}}^i e_{\bar{c}}^j \partial_i e_{ja}] + e_{[\bar{c}}^i e_a^j \partial_i e_{j\bar{b}]} . \quad (2.16)$$

Similarly, one finds

$$\omega_{\bar{a}bc} = \omega_{\bar{a}bc}^L(e) + \frac{1}{2} H_{\bar{a}bc}, \quad (2.17)$$

where we flattened the indices of H as in (2.15).³

For the ‘diagonal’ components, having either only unbarred or barred indices, the totally antisymmetric parts are determined by (2.11) as follows

$$\omega_{[abc]} = -\frac{1}{3} \Omega_{[abc]} = -f_{[abc]}, \quad \omega_{[\bar{a}\bar{b}\bar{c}]} = -\frac{1}{3} \Omega_{[\bar{a}\bar{b}\bar{c}]} = -f_{[\bar{a}\bar{b}\bar{c}]} . \quad (2.18)$$

Again, we may determine these connections for the gauge choice (2.7) and $\tilde{\partial}^i = 0$. One finds,

$$\omega_{[abc]} = \omega_{[abc]}^L(e) + \frac{1}{6} H_{abc}, \quad \omega_{[\bar{a}\bar{b}\bar{c}]} = -\omega_{[\bar{a}\bar{b}\bar{c}]}^L(e) + \frac{1}{6} H_{\bar{a}\bar{b}\bar{c}}, \quad (2.19)$$

where we flattened the indices on H .

The torsion constraint leaves the mixed Young tableaux representation in ω_{abc} and $\omega_{\bar{a}\bar{b}\bar{c}}$ undetermined, but its trace part can be fixed by imposing a covariant constraint that allows for partial integration in presence of the dilaton density,

$$\int e^{-2d} V \nabla_A V^A = - \int e^{-2d} V^A \nabla_A V, \quad (2.20)$$

for arbitrary V and V^A . This implies

$$\omega_{BA}{}^B = -\tilde{\Omega}_A \equiv -\sqrt{2} e^{2d} \partial_M (E_A{}^M e^{-2d}), \quad (2.21)$$

where we introduced $\tilde{\Omega}_A$ for later use. Note that this determines precisely $\omega_{ba}{}^b$ and $\omega_{\bar{b}\bar{a}}{}^{\bar{b}}$, because the last two indices cannot be mixed.

³We note that the relative sign between $\omega_{a\bar{b}\bar{c}}$ and $\omega_{a\bar{b}\bar{c}}^L$ in (2.15) is due to the fact that we lower barred indices with $\mathcal{G}_{\bar{a}\bar{b}} = -\eta_{\bar{a}\bar{b}}$, see eq. (2.6), while in the standard expression (2.16) for the spin connection the index is lowered with $\eta_{\bar{a}\bar{b}}$. Correspondingly, there is no relative sign in (2.17) because here indices are lowered with $\mathcal{G}_{ab} = \eta_{ab}$.

Finally, we can introduce an invariant scalar curvature and Ricci tensor. In the frame formalism there is an invariant curvature tensor \mathcal{R}_{ABCD} , but it is generally not a function of the determined connections only. For the derived curvature scalar and Ricci tensor, however, it depends only on the determined connections. Without repeating the details of the construction, we give the explicit expressions.

The scalar curvature can be defined as the trace over, say, barred indices as follows

$$\begin{aligned}\mathcal{R} &\equiv -\mathcal{R}_{\bar{a}\bar{b}}^{\bar{a}\bar{b}} = -2E_{\bar{a}}\omega_{\bar{b}}^{\bar{a}\bar{b}} - \frac{3}{2}\omega_{[\bar{a}\bar{b}\bar{c}]}^{\bar{a}\bar{b}\bar{c}}\omega^{[\bar{a}\bar{b}\bar{c}]} + \omega_{\bar{a}}^{\bar{c}\bar{a}}\omega_{\bar{b}\bar{c}}^{\bar{b}} - \frac{1}{2}\omega_{\bar{a}\bar{b}\bar{c}}\omega^{\bar{a}\bar{b}\bar{c}} \\ &= 2E_{\bar{a}}\tilde{\Omega}^{\bar{a}} + \tilde{\Omega}_{\bar{a}}^2 - \frac{1}{2}\Omega_{\bar{a}\bar{b}\bar{c}}^2 - \frac{1}{6}\Omega_{[\bar{a}\bar{b}\bar{c}]}^2,\end{aligned}\tag{2.22}$$

where we have written in the second line the explicit expression in terms of Ω and thereby in terms of the physical fields. The Ricci tensor reads

$$\mathcal{R}_{a\bar{b}} = E_{\bar{c}}\omega_{a\bar{b}}^{\bar{c}} - E_a\omega_{\bar{c}\bar{b}}^{\bar{c}} + \omega_{d\bar{b}}^{\bar{c}}\omega_{\bar{c}a}^d - \omega_{a\bar{b}}^{\bar{d}}\omega_{\bar{c}d}^{\bar{c}}.\tag{2.23}$$

These curvature invariants can be obtained by variation of the (bosonic) double field theory action. In order to see this it is convenient to introduce the variation

$$\Delta E_{AB} := E_B^M \delta E_{AM},\tag{2.24}$$

which is antisymmetric in A, B as a consequence of the constancy of \mathcal{G}_{AB} . Under the local $O(1,9)_L \times O(1,9)_R$ this variation reads $\Delta E_{ab} = \Lambda_{ab}$ and $\Delta E_{\bar{a}\bar{b}} = \Lambda_{\bar{a}\bar{b}}$. Thus, only the off-diagonal variation is not pure-gauge and the corresponding general variation of the action (1.3) can be written in terms of the curvatures as [7]

$$\delta S = -2 \int dx d\tilde{x} e^{-2d} \left(\delta d \mathcal{R} + \Delta E_{a\bar{b}} \mathcal{R}^{a\bar{b}} \right),\tag{2.25}$$

which will be used below.

2.2 $\mathcal{N} = 1$ Double Field Theory

We give now the $\mathcal{N} = 1$ supersymmetric extension of double field theory in the frame formulation reviewed above. The fermionic fields are the ‘gravitino’ ψ_a and the ‘dilatino’ ρ , and we will later see how they are related to the conventional gravitino and dilatino via a field redefinition. These fields are scalars under $O(10,10)$ and generalized diffeomorphisms and, together with the $\mathcal{N} = 1$ supersymmetry parameter ϵ , transform under the local $O(1,9)_L \times O(1,9)_R$ as follows

$$\begin{aligned}\Psi_a &: \quad \text{vector of } O(1,9)_L, \text{ spinor of } O(1,9)_R, \\ \rho &: \quad \text{spinor of } O(1,9)_R, \\ \epsilon &: \quad \text{spinor of } O(1,9)_R.\end{aligned}\tag{2.26}$$

The $\mathcal{N} = 1$ supersymmetric extension of (1.3) is given by (1.4),

$$S_{\mathcal{N}=1} = \int dx d\tilde{x} e^{-2d} \left(\mathcal{R}(E, d) - \bar{\Psi}^a \gamma^{\bar{b}} \nabla_{\bar{b}} \Psi_a + \bar{\rho} \gamma^{\bar{a}} \nabla_{\bar{a}} \rho + 2\bar{\Psi}^a \nabla_a \rho \right),\tag{2.27}$$

where all covariant derivatives are with respect to the connections introduced above. We will see below that in here and in the supersymmetry rules all undetermined connections drop out. When acting on $O(1,9)_R$ spinors the covariant derivatives are given by

$$\nabla_a = E_a - \frac{1}{4}\omega_{a\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}, \quad \nabla_{\bar{a}} = E_{\bar{a}} - \frac{1}{4}\omega_{\bar{a}b\bar{c}}\gamma^{\bar{b}\bar{c}}. \quad (2.28)$$

We observe that (2.27) is manifestly $O(1,9)_L \times O(1,9)_R$ invariant, because unbarred and barred indices are properly contracted, and the $\gamma^{\bar{a}}$ are gamma matrices of $O(1,9)_R$, so that all suppressed spinor indices belong to $O(1,9)_R$. More precisely, we define the $\gamma^{\bar{a}}$ to satisfy

$$\{\gamma^{\bar{a}}, \gamma^{\bar{b}}\} = -2\mathcal{G}^{\bar{a}\bar{b}} = 2\eta^{\bar{a}\bar{b}}, \quad (2.29)$$

where the signs are such that the $\gamma^{\bar{a}}$ can be chosen to be conventional gamma matrices in ten dimensions. We note that, according to our convention, on $\gamma_{\bar{a}}$ the index is lowered with $\mathcal{G}_{\bar{a}\bar{b}} = -\eta_{\bar{a}\bar{b}}$ so that it differs from the conventional ten-dimensional gamma matrix with a lower index by a sign. Similarly, the minus signs in (2.28) are due to the lowering of indices on $\omega_{A\bar{b}\bar{c}}$ with $\mathcal{G}_{\bar{a}\bar{b}}$. Let us finally stress that the assignment (2.26) of $O(1,9)_L \times O(1,9)_R$ representations is related to the constraint (2.6). We could have chosen the opposite signatures for \mathcal{G}_{ab} and $\mathcal{G}_{\bar{a}\bar{b}}$, but then supersymmetry would require the gravitino to be a vector under $O(1,9)_R$ and a spinor under $O(1,9)_L$.

The action (2.27) is manifestly invariant under generalized diffeomorphisms,

$$\begin{aligned} \delta_\xi E_A^M &= \widehat{\mathcal{L}}_\xi E_A^M, & \delta_\xi d &= \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M, \\ \delta_\xi \Psi_a &= \xi^M \partial_M \Psi_a, & \delta_\xi \rho &= \xi^M \partial_M \rho, \end{aligned} \quad (2.30)$$

because with the fermions transforming as scalars the (flattened) derivatives in (2.27) transform covariantly. In addition, the action is invariant under the $\mathcal{N} = 1$ supersymmetry transformations [25]

$$\begin{aligned} \Delta_\epsilon E_{a\bar{b}} &= -\frac{1}{2}\bar{\epsilon}\gamma_{\bar{b}}\Psi_a, & \delta_\epsilon d &= -\frac{1}{4}\bar{\epsilon}\rho, \\ \delta_\epsilon \Psi_a &= \nabla_a \epsilon, & \delta_\epsilon \rho &= \gamma^{\bar{a}}\nabla_{\bar{a}}\epsilon. \end{aligned} \quad (2.31)$$

Here, we have written the transformation of the frame field in terms of the variation (2.24). Due to the $O(1,9)_L \times O(1,9)_R$ gauge freedom, we can assume for the diagonal supersymmetry variations $\Delta_\epsilon E_{ab} = \Delta_\epsilon E_{\bar{a}\bar{b}} = 0$.

Let us now verify that (2.27) is invariant under (2.31), again up to higher-order fermi terms. We start with the variation of the bosonic part, which can be obtained directly by inserting the fermionic supersymmetry rules of (2.31) into (2.25),

$$e^{2d}\delta_\epsilon \mathcal{L}_B = \frac{1}{2}\bar{\epsilon}\rho\mathcal{R} + \bar{\epsilon}\gamma_{\bar{b}}\Psi_a\mathcal{R}^{a\bar{b}}, \quad (2.32)$$

where we denoted the bosonic Lagrangian by \mathcal{L}_B . Denoting the fermionic part similarly by \mathcal{L}_F , one finds

$$\begin{aligned} e^{2d}\delta_\epsilon \mathcal{L}_F &= -2\bar{\Psi}^a\gamma^{\bar{b}}\nabla_{\bar{b}}\nabla_a\epsilon + 2\bar{\rho}\gamma^{\bar{a}}\nabla_{\bar{a}}(\gamma^{\bar{b}}\nabla_{\bar{b}}\epsilon) + 2\nabla^a\bar{\epsilon}\nabla_a\rho + 2\bar{\Psi}^a\nabla_a(\gamma^{\bar{b}}\nabla_{\bar{b}}\epsilon) \\ &= -2\bar{\Psi}^a[\gamma^{\bar{b}}\nabla_{\bar{b}}, \nabla_a]\epsilon + 2\bar{\rho}\left(\gamma^{\bar{a}}\nabla_{\bar{a}}\gamma^{\bar{b}}\nabla_{\bar{b}} - \nabla^a\nabla_a\right)\epsilon. \end{aligned} \quad (2.33)$$

Here we have used that according to (2.20) the covariant derivatives allow us to freely partially integrate in presence of the dilaton density. Moreover, in the second line we have combined the first and last and the second and third term. We can now use the identities [25]

$$\begin{aligned} \left(\gamma^{\bar{a}} \nabla_{\bar{a}} \gamma^{\bar{b}} \nabla_{\bar{b}} - \nabla^a \nabla_a \right) \epsilon &= -\frac{1}{4} \mathcal{R} \epsilon, \\ \left[\gamma^{\bar{b}} \nabla_{\bar{b}}, \nabla_a \right] \epsilon &= -\frac{1}{2} \gamma^{\bar{b}} \mathcal{R}_{a\bar{b}} \epsilon, \end{aligned} \quad (2.34)$$

which will be proved in the appendix, to see that this cancels precisely the variation (2.32) of the bosonic term, proving supersymmetric invariance.

We turn now to the closure of the supersymmetry transformations. Since these are an invariance of the action (2.27) they must close into the other local symmetries of the theory, which are generalized diffeomorphisms and the doubled local Lorentz transformations $O(1,9)_L \times O(1,9)_R$. It is instructive, however, to investigate this explicitly, and so we verify in the following closure on the bosonic fields. For the dilaton we compute

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] d = \frac{1}{4} (\bar{\epsilon}_1 \gamma^{\bar{a}} \nabla_{\bar{a}} \epsilon_2 - \bar{\epsilon}_2 \gamma^{\bar{a}} \nabla_{\bar{a}} \epsilon_1) = \frac{1}{4} \bar{\epsilon}_1 \gamma^{\bar{a}} (E_{\bar{a}} - \frac{1}{4} \omega_{\bar{a}\bar{b}\bar{c}} \gamma^{\bar{b}\bar{c}}) \epsilon_2 - (1 \leftrightarrow 2). \quad (2.35)$$

Let us work out the first term in here,

$$\frac{1}{4} \bar{\epsilon}_1 \gamma^{\bar{a}} E_{\bar{a}} \epsilon_2 - (1 \leftrightarrow 2) = \frac{\sqrt{2}}{4} \bar{\epsilon}_1 \gamma^{\bar{a}} E_{\bar{a}}^M \partial_M \epsilon_2 - (1 \leftrightarrow 2) = \frac{1}{2\sqrt{2}} E_{\bar{a}}^M \partial_M (\bar{\epsilon}_1 \gamma^{\bar{a}} \epsilon_2), \quad (2.36)$$

using $\bar{\epsilon}_1 \gamma^{\bar{a}} \epsilon_2 = -\bar{\epsilon}_2 \gamma^{\bar{a}} \epsilon_1$. For the second term we compute

$$-\frac{1}{16} \omega_{\bar{a}\bar{b}\bar{c}} \bar{\epsilon}_1 \gamma^{\bar{a}} \gamma^{\bar{b}\bar{c}} \epsilon_2 - (1 \leftrightarrow 2) = -\frac{1}{16} \omega_{\bar{a}\bar{b}\bar{c}} \bar{\epsilon}_1 (\gamma^{\bar{a}\bar{b}\bar{c}} - 2\mathcal{G}^{\bar{a}[\bar{b}} \gamma^{\bar{c}]}) \epsilon_2 - (1 \leftrightarrow 2). \quad (2.37)$$

The first term in here vanishes due to the antisymmetrization in $(1 \leftrightarrow 2)$ and $\bar{\epsilon}_1 \gamma^{\bar{a}\bar{b}\bar{c}} \epsilon_2 = \bar{\epsilon}_2 \gamma^{\bar{a}\bar{b}\bar{c}} \epsilon_1$. The second term gives with (2.21)

$$-\frac{1}{4} \omega_{\bar{a}\bar{b}\bar{c}} \bar{\epsilon}_1 \gamma^{\bar{c}} \epsilon_2 = \frac{1}{2\sqrt{2}} (\partial_M E_{\bar{c}}^M - 2E_{\bar{c}}^M \partial_M d) \bar{\epsilon}_1 \gamma^{\bar{c}} \epsilon_2. \quad (2.38)$$

The first term in here combines with (2.36) to give $\frac{1}{2\sqrt{2}} \partial_M (E_{\bar{c}}^M \bar{\epsilon}_1 \gamma^{\bar{c}} \epsilon_2)$. The second term takes the form of a transport term so that we have shown in total

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] d = \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M, \quad \xi^M = -\frac{1}{\sqrt{2}} E_{\bar{a}}^M \bar{\epsilon}_1 \gamma^{\bar{a}} \epsilon_2. \quad (2.39)$$

Thus, the supersymmetry transformations close into generalized diffeomorphisms, as required.

Next, we verify closure on E_A^M . We compute

$$\begin{aligned} [\delta_{\epsilon_1}, \delta_{\epsilon_2}] E_{aM} &= \delta_{\epsilon_1} (E_M^B E_B^N \delta_{\epsilon_2} E_{aN}) - (1 \leftrightarrow 2) \\ &= \delta_{\epsilon_1} (E_M^{\bar{b}} \Delta_{\epsilon_2} E_{a\bar{b}}) - (1 \leftrightarrow 2) = -\frac{1}{2} \delta_{\epsilon_1} (E_M^{\bar{c}} \bar{\epsilon}_2 \gamma_{\bar{c}} \Psi_a) - (1 \leftrightarrow 2), \end{aligned} \quad (2.40)$$

where we used that we can set $\Delta_{\epsilon} E_{ab} = 0$ by an appropriate $O(1,9)_L$ transformation, and we relabeled an index in the last equality. In order to disentangle the generalized diffeomorphisms

and local $O(1,9)_L \times O(1,9)_R$ transformations we project (2.40) by multiplying with E_b^M and E_b^M , respectively. For the first we obtain

$$\begin{aligned} E_b^M [\delta_{\epsilon_1}, \delta_{\epsilon_2}] E_{aM} &= -\frac{1}{2} E_b^M E_M^{\bar{c}} \bar{\epsilon}_2 \gamma_{\bar{c}} \nabla_a \epsilon_1 - (1 \leftrightarrow 2) \\ &= -\frac{1}{2} \bar{\epsilon}_2 \gamma_{\bar{b}} (\sqrt{2} E_a^N \partial_N - \frac{1}{4} \omega_{a\bar{c}\bar{d}} \gamma^{\bar{c}\bar{d}}) \epsilon_1 - (1 \leftrightarrow 2), \end{aligned} \quad (2.41)$$

where we used that only the variation of Ψ_a is non-trivial as a consequence of $\Delta_\epsilon E_{\bar{a}\bar{b}} = 0$. The first term in here reads

$$-\frac{1}{\sqrt{2}} (\bar{\epsilon}_2 \gamma_{\bar{b}} \partial_N \epsilon_1 - \bar{\epsilon}_1 \gamma_{\bar{b}} \partial_N \epsilon_2) E_a^N = \frac{1}{\sqrt{2}} \partial_N (\bar{\epsilon}_1 \gamma_{\bar{b}} \epsilon_2) E_a^N. \quad (2.42)$$

For the second term we use as above that the $\gamma^{(3)}$ structure drops due to the antisymmetrization in $(1 \leftrightarrow 2)$. The remaining structure proportional to $\gamma^{(1)}$ is then automatically antisymmetric in $(1 \leftrightarrow 2)$ and thus reads

$$-\frac{1}{2} \omega_{a\bar{c}\bar{d}} \bar{\epsilon}_2 \delta_{\bar{b}}^{[\bar{c}} \gamma^{\bar{d}]} \epsilon_1 = \frac{1}{2} \omega_{a\bar{b}\bar{c}} \bar{\epsilon}_1 \gamma^{\bar{c}} \epsilon_2. \quad (2.43)$$

The spin connection is given by

$$\omega_{a\bar{b}\bar{c}} = -3f_{[a\bar{b}\bar{c}]} = \sqrt{2} (E_a^K E_b^N \partial_K E_{\bar{c}N} - E_b^K \partial_K E_{\bar{c}}^N E_{aN} - E_{\bar{c}}^K E_b^N \partial_K E_{aN}). \quad (2.44)$$

Inserting this into (2.43) and combining with (2.42) we obtain in total

$$E_b^M [\delta_{\epsilon_1}, \delta_{\epsilon_2}] E_{aM} = E_b^M (\xi^N \partial_N E_{aM} + (\partial_M \xi^N - \partial^N \xi_M) E_{aN}), \quad (2.45)$$

where

$$\xi^M = -\frac{1}{\sqrt{2}} E_a^M \bar{\epsilon}_1 \gamma^{\bar{a}} \epsilon_2, \quad (2.46)$$

is the same parameter as in (2.39).

Next, we turn to the other projection,

$$\begin{aligned} E_b^M [\delta_{\epsilon_1}, \delta_{\epsilon_2}] E_{aM} &= -\frac{1}{2} E_b^M \delta_{\epsilon_1} E_M^{\bar{c}} \bar{\epsilon}_2 \gamma_{\bar{c}} \Psi_a - (1 \leftrightarrow 2) \\ &= \frac{1}{2} \Delta_{\epsilon_1} E_{b\bar{c}} \bar{\epsilon}_2 \gamma^{\bar{c}} \Psi_a - (1 \leftrightarrow 2) = \frac{1}{2} (\bar{\epsilon}_1 \gamma_{\bar{c}} \Psi_{[a}) (\bar{\epsilon}_2 \gamma^{\bar{c}} \Psi_{b]}) . \end{aligned} \quad (2.47)$$

The last term is antisymmetric in a, b and can thus be interpreted as a field-dependent $O(1,9)_L$ gauge transformation. Here we would have expected also a generalized diffeomorphism with parameter (2.46), but for this particular projection such a term can actually be absorbed into an $O(1,9)_L$ gauge transformation. To show this it suffices to note that by definition (2.2)

$$E_b^M \delta_\xi E_{aM} = \xi^N E_b^M \partial_N E_{aM} - 2\partial_M \xi_N E_{[a}^M E_{b]}^N, \quad (2.48)$$

is antisymmetric in a, b . Thus, equivalently, (2.47) closes into the required generalized diffeomorphisms and into local $O(1,9)_L$ transformations with parameter

$$\Lambda_{ab} = \frac{1}{2} (\bar{\epsilon}_1 \gamma_{\bar{c}} \Psi_{[a}) (\bar{\epsilon}_2 \gamma^{\bar{c}} \Psi_{b]}) + \xi^N E_{[a}^M \partial_N E_{b]M} + 2\partial_M \xi_N E_{[a}^M E_{b]}^N, \quad (2.49)$$

with ξ^M given by (2.46). In total, combining (2.45) and (2.47), we have verified closure,

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] E_{aM} = \widehat{\mathcal{L}}_\xi E_{aM} + \Lambda_a{}^b E_{bM} , \quad (2.50)$$

with parameters given by (2.46) and (2.49). The verification for $E_{\bar{a}M}$ is completely analogous. In particular, the corresponding $O(1, 9)_R$ parameter is given by

$$\Lambda_{\bar{a}\bar{b}} = \frac{1}{2}(\bar{\epsilon}_1 \gamma_{[\bar{a}} \Psi^c)(\bar{\epsilon}_2 \gamma_{\bar{b}]} \Psi_c) + \xi^N E_{[\bar{a}}{}^M \partial_N E_{\bar{b}]M} + 2\partial_M \xi_N E_{[\bar{a}}{}^M E_{\bar{b}]}{}^N . \quad (2.51)$$

In general, the supersymmetry transformations close according to

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_\xi + \delta_\Lambda + \delta_{\bar{\Lambda}} , \quad (2.52)$$

with ξ given by (2.46), Λ by (2.49) and $\bar{\Lambda}$ by (2.51). We finally note that even though we have not employed the field equations for the above computation, in general the gauge algebra (2.52) will only hold on-shell. In fact, without auxiliary fields supersymmetry transformations close on the fermions only modulo their field equations. In contrast, for the bosons the field equations do not enter on dimensional grounds, because they are second-order in derivatives.

2.3 Reduction to standard $\mathcal{N} = 1$ supergravity

Let us now verify that the action (2.27) and the supersymmetry rules (2.31) reduce to the conventional $\mathcal{N} = 1$ supergravity in $D = 10$ upon setting $\tilde{\partial}^i = 0$. As discussed above, this comparison requires a partial gauge fixing of the local $O(1, 9)_L \times O(1, 9)_R$ to the diagonal subgroup. We can then write the frame field as in (2.7) in terms of b_{ij} and the conventional vielbein $e_i{}^a$. In the following we will show that the conventional $\mathcal{N} = 1$ theory is related to the action following from (2.27) by a field redefinition.

We start by recalling minimal $\mathcal{N} = 1$, $D = 10$ supergravity in the string frame. The field content is given by

$$(e_i{}^a, b_{ij}, \phi, \psi_i, \lambda) , \quad (2.53)$$

where the fermionic fields are the gravitino ψ_i and the dilatino λ . The action reads⁴

$$\begin{aligned} S = \int d^{10}x e e^{-2\phi} & \left[\left(R + 4\partial^i \phi \partial_i \phi - \frac{1}{12} H^{ijk} H_{ijk} \right) \right. \\ & - \bar{\psi}_i \gamma^{ijk} D_j \psi_k + 2\bar{\psi}^i (\partial_i \phi) \gamma^j \psi_j - 2\bar{\lambda} \gamma^i D_i \lambda - \bar{\psi}_i (\not{\partial} \phi) \gamma^i \lambda \\ & \left. + \frac{1}{24} H_{ijk} \left(\bar{\psi}_m \gamma^{mijk} \psi_n + 6\bar{\psi}^i \gamma^j \psi^k - 2\bar{\psi}_m \gamma^{ijk} \gamma^m \lambda \right) \right] , \end{aligned} \quad (2.54)$$

where $H_{ijk} = 3\partial_{[i} b_{jk]}$ and $e = \det(e_i{}^a)$. Here, we denoted the covariant derivatives with respect to the standard torsion-free Levi-Civita connection by D_i in order to distinguish them from the covariant derivatives ∇ with respect to Siegel's connections. If a non-trivial connection, say $\hat{\omega}$, is used this will be indicated explicitly as $D_i(\hat{\omega})$. We stress that the spin connection defining the Ricci scalar and thus the Einstein-Hilbert term is also the conventional torsion-free connection rather than the super-covariant one. We will not take into account terms higher

⁴This form of the supergravity action is $\frac{1}{2}$ times the one obtained from eq. (10) of [27] by performing the redefinitions $\phi^{-\frac{3}{2}} \rightarrow e^{-\phi}$, $\lambda \rightarrow \sqrt{2}\lambda$, $F_{ijk} \rightarrow \frac{1}{3\sqrt{2}}H_{ijk}$, $B_{ij} \rightarrow \frac{1}{\sqrt{2}}b_{ij}$.

order in fermions. Up to this order, the supersymmetry transformations leaving (2.54) invariant read

$$\begin{aligned}
\delta_\epsilon e_i^a &= \frac{1}{2} \bar{\epsilon} \gamma^a \psi_i - \frac{1}{4} \bar{\epsilon} \lambda e_i^a , \\
\delta_\epsilon \phi &= -\bar{\epsilon} \lambda , \\
\delta_\epsilon \psi_i &= D_i \epsilon - \frac{1}{8} \gamma_i (\not{\partial} \phi) \epsilon + \frac{1}{96} (\gamma_i^{klm} - 9 \delta_i^k \gamma^{lm}) H_{klm} \epsilon , \\
\delta_\epsilon \lambda &= -\frac{1}{4} (\not{\partial} \phi) \epsilon + \frac{1}{48} \gamma^{ijk} H_{ijk} \epsilon , \\
\delta_\epsilon b_{ij} &= \frac{1}{2} (\bar{\epsilon} \gamma_i \psi_j - \bar{\epsilon} \gamma_j \psi_i) - \frac{1}{2} \bar{\epsilon} \gamma_{ij} \lambda .
\end{aligned} \tag{2.55}$$

Next, we perform some field redefinitions that are necessary in order to compare with the double field theory variables [25],

$$\Psi_i \equiv \psi_i - \frac{1}{2} \gamma_i \lambda , \quad \rho \equiv \gamma^i \psi_i - \lambda = \gamma^i \Psi_i + 4\lambda . \tag{2.56}$$

Moreover, as usual we introduce the T-duality invariant dilaton $e^{-2d} = e e^{-2\phi}$. Written in terms of these variables, the action (2.54) reads

$$\begin{aligned}
S = \int d^{10}x e^{-2d} & \left[\left(R + 4 \partial^i \phi \partial_i \phi - \frac{1}{12} H^{ijk} H_{ijk} \right) - \bar{\Psi}^j \gamma^i D_i \Psi_j + 2 \bar{\Psi}^i D_i \rho \right. \\
& \left. + \bar{\rho} \gamma^i D_i \rho + \frac{1}{4} \bar{\Psi}^i \not{H} \Psi_i - \frac{1}{4} \bar{\rho} \not{H} \rho + \frac{1}{2} H_{ijk} \bar{\Psi}^i \gamma^j \Psi^k + \frac{1}{4} H_{ijk} \bar{\rho} \gamma^{ij} \Psi^k \right] ,
\end{aligned} \tag{2.57}$$

where $\not{H} = \frac{1}{3!} \gamma^{ijk} H_{ijk}$. This is the final form of the action that is suitable for the comparison with double field theory. The supersymmetry variations written in terms of (2.56) are

$$\begin{aligned}
\delta_\epsilon e_i^a &= \frac{1}{2} \bar{\epsilon} \gamma^a \Psi_i , \\
\delta_\epsilon b_{ij} &= \bar{\epsilon} \gamma_{[i} \Psi_{j]} , \\
\delta_\epsilon d &= -\frac{1}{4} \bar{\epsilon} \rho , \\
\delta_\epsilon \Psi_i &= D_i (\hat{\omega}) \epsilon , \\
\delta_\epsilon \rho &= \gamma^i D_i \epsilon - \frac{1}{24} H_{ijk} \gamma^{ijk} \epsilon - (\not{\partial} \phi) \epsilon ,
\end{aligned} \tag{2.58}$$

where we introduced a redefinition of the Levi-Civita spin connection ω^L ,

$$\hat{\omega}_{abc} = \omega_{abc}^L - \frac{1}{2} H_{abc} , \tag{2.59}$$

because this is the combination that appears naturally in double field theory, see (2.15).

Let us now return to the double field theory action and supersymmetry transformations (2.27) and (2.31). We first observe that the kinetic terms in (2.27) and (2.57) agree, upon converting flat into curved indices. We will show next that the extra terms in the action (2.57) and the supersymmetry rules (2.58) as compared to double field theory are precisely reproduced by the non-trivial connections inside the covariant derivatives in double field theory.

We start with the supersymmetry transformations. First we note that the variation of ψ_i agrees with the double field theory variation (2.31), because (2.59) coincides with (2.15). Next, consider the variation of the dilatino ρ in (2.31), which reads

$$\delta_\epsilon \rho = \gamma^{\bar{a}} \nabla_{\bar{a}} \epsilon = \gamma^{\bar{a}} \left(E_{\bar{a}} - \frac{1}{4} \omega_{\bar{a}\bar{b}\bar{c}} \gamma^{\bar{b}\bar{c}} \right) \epsilon. \quad (2.60)$$

We can now work out the connection term in here,

$$\omega_{\bar{a}\bar{b}\bar{c}} \gamma^{\bar{a}} \gamma^{\bar{b}\bar{c}} = \omega_{\bar{a}\bar{b}\bar{c}} (\gamma^{\bar{a}\bar{b}\bar{c}} - \mathcal{G}^{\bar{a}\bar{b}} \gamma^{\bar{c}} + \mathcal{G}^{\bar{a}\bar{c}} \gamma^{\bar{b}}) = \omega_{[\bar{a}\bar{b}\bar{c}]} \gamma^{\bar{a}\bar{b}\bar{c}} + 2\omega_{\bar{a}\bar{b}}^{\bar{a}} \gamma^{\bar{b}}, \quad (2.61)$$

where we used that ω is antisymmetric in its last two indices. Insertion into (2.60) then yields

$$\delta_\epsilon \rho = \left(\gamma^{\bar{a}} E_{\bar{a}} - \frac{1}{4} \omega_{[\bar{a}\bar{b}\bar{c}]} \gamma^{\bar{a}\bar{b}\bar{c}} - \frac{1}{2} \omega_{\bar{a}\bar{b}}^{\bar{a}} \gamma^{\bar{b}} \right) \epsilon. \quad (2.62)$$

We see that only the totally antisymmetric and trace parts of the connections enter, which in turn are fully determined by the constraints. This observation, which has first been made in [25], will be used repeatedly below. Inserting now (2.19) and (2.21) for these determined connections we can rewrite (2.60) as

$$\delta_\epsilon \rho = \gamma^i D_i \epsilon - \frac{1}{24} H_{ijk} \gamma^{ijk} \epsilon - (\not{\partial} \phi) \epsilon, \quad (2.63)$$

which agrees with the required supersymmetry variation of ρ in (2.58). Thus, we have shown that the supersymmetry variations of the fermions in double field theory reproduce the transformations required by $\mathcal{N} = 1$ supergravity. For the supersymmetry variations of the bosonic fields consistency with double field theory is manifest for the dilaton d , while for the metric and b -field a short computation is required: variation of (2.7) yields

$$\Delta_\epsilon E_{\bar{a}\bar{b}} = e_b^i \delta_\epsilon e_{ia} + e_a^i \delta_\epsilon e_{i\bar{b}} - \frac{1}{2} e_a^i e_b^j \delta_\epsilon b_{ij} = -\frac{1}{2} \bar{\epsilon} \gamma_{\bar{b}} \Psi_a. \quad (2.64)$$

Due to the relative sign in the contraction of barred indices discussed after eq. (2.29) we have to identify $\gamma_i = -e_i^{\bar{a}} \gamma_{\bar{a}}$. Projecting (2.64) onto its antisymmetric part we then read off $\delta_\epsilon b_{ij} = \bar{\epsilon} \gamma_{[i} \Psi_{j]}$, in precise agreement with (2.58). In addition, the symmetric projection of (2.64) determines the symmetric part of the supersymmetry variation $e_b^i \delta_\epsilon e_{ia}$. Its antisymmetric part is undetermined, as it should be, because this freedom reflects the diagonal local Lorentz group that is left unbroken by the gauge-fixed form (2.7). It is then easy to see that, up to these local Lorentz transformations, (2.64) yields $\delta_\epsilon e_i^a$ as in (2.58). In total, the supersymmetry transformations of double field theory reduce precisely to (2.58).

We turn now to the action. Similarly to the discussion of the supersymmetry transformations it is easy to see that all connections are determined and that writing them out in terms of the Levi-Civita connection reproduces the H -dependent terms in (2.57).

Let us start with the covariant derivative $\nabla_{\bar{b}}$ in the first fermionic term in (2.27), which acts on Ψ_a as an $O(1,9)_R$ spinor and as an $O(1,9)_L$ vector, i.e.,

$$-\bar{\Psi}^a \gamma^{\bar{b}} \nabla_{\bar{b}} \Psi_a = -\bar{\Psi}^a \gamma^{\bar{b}} \left(E_{\bar{b}} \Psi_a - \frac{1}{4} \omega_{\bar{b}\bar{c}\bar{d}} \gamma^{\bar{c}\bar{d}} \Psi_a + \omega_{\bar{b}a}^{\bar{c}} \Psi_c \right). \quad (2.65)$$

As in (2.63), the terms combine into $-\bar{\Psi}^j \gamma^i D_i \Psi_j$ and $\frac{1}{4} \bar{\Psi}^i \not{H} \Psi_i$, while a d -dependent term drops out as a consequence of $\Psi^j \gamma^i \Psi_j = 0$. The last term in (2.65) gives in addition to the standard spin connection an extra contribution,

$$-\bar{\Psi}^a \gamma^{\bar{b}} \omega_{\bar{b}a}^{c} \Psi_c = -\bar{\Psi}^a \gamma^{\bar{b}} \left(\omega_{\bar{b}a}^{\text{L}c} + \frac{1}{2} H_{\bar{b}a}^{c} \right) \Psi_c = -\bar{\Psi}^a \gamma^{\bar{b}} \omega_{\bar{b}a}^{\text{L}c} \Psi_c + \frac{1}{2} H_{\bar{a}\bar{b}c} \bar{\Psi}^a \gamma^{\bar{b}} \Psi^c, \quad (2.66)$$

reproducing the term $\frac{1}{2} H_{ijk} \bar{\Psi}^i \gamma^j \Psi^k$ in (2.57).

Next, we consider the kinetic term of ρ which as in (2.63) reduces to

$$\bar{\rho} \gamma^{\bar{a}} \nabla_{\bar{a}} \rho = \bar{\rho} \gamma^i D_i \rho - \frac{1}{24} \bar{\rho} H_{ijk} \gamma^{ijk} \rho. \quad (2.67)$$

Finally, the last structure in (2.27) yields

$$2\bar{\Psi}^a \nabla_a \rho = 2\bar{\Psi}^a \left(E_a \rho - \frac{1}{4} \omega_{a\bar{b}\bar{c}} \gamma^{\bar{b}\bar{c}} \rho \right) = 2\bar{\Psi}^i D_i \rho + \frac{1}{4} H_{ijk} \bar{\rho} \gamma^{ij} \Psi^k. \quad (2.68)$$

Collecting the term $\frac{1}{4} \bar{\Psi}^i \not{H} \Psi_i$ originating from (2.65) together with (2.66), (2.67) and (2.68) we infer that the double field theory action reproduces (2.57). Summarizing, we have shown that the $\mathcal{N} = 1$ supersymmetric double field theory reduces for $\tilde{\partial}^i = 0$ to minimal $\mathcal{N} = 1$ supergravity in $D = 10$.

3 Heterotic Supersymmetric Double Field Theory

In this section we extend the above construction to the coupling of an arbitrary number n of abelian vector multiplets. For $n = 16$ this completes the construction of [9] by the fermionic or NS-R sector of heterotic superstring theory truncated to the Cartan subalgebra of $E_8 \times E_8$ or $SO(32)$. We first review the extension of the frame formalism, in which the tangent space group is extended to $O(1, 9 + n) \times O(1, 9)$. Then we show that the same $\mathcal{N} = 1$ double field theory action (1.4), but interpreted with respect to the enlarged frame and spinor fields, reduces to $\mathcal{N} = 1$ supergravity coupled to n vector multiplets upon setting the extra derivatives to zero.

3.1 $\mathcal{N} = 1$ double field theory with local $O(1, 9 + n) \times O(1, 9)$ symmetry

Let us begin by reviewing the double field theory formulation in presence of n abelian gauge vectors $A_i{}^\alpha$ [9]. The generalized metric is extended to an $O(10 + n, 10)$ group element, naturally encoding these additional fields. Correspondingly, there are $20 + n$ coordinates,

$$X^M = (\tilde{x}_i, y^\alpha, x^i), \quad \partial_M = (\tilde{\partial}^i, \partial_\alpha, \partial_i), \quad (3.1)$$

transforming as an $O(10 + n, 10)$ vector, with indices that are raised and lowered with

$$\eta_{MN} = \begin{pmatrix} 0 & 0 & \mathbf{1}_{10} \\ 0 & \mathbf{1}_n & 0 \\ \mathbf{1}_{10} & 0 & 0 \end{pmatrix}. \quad (3.2)$$

We still impose the constraint $\eta^{MN} \partial_M \partial_N = 0$, using the $O(10 + n, 10)$ invariant metric (3.2). It implies that one can always rotate into a frame in which $\tilde{\partial}^i = \partial_\alpha = 0$.

Next, we can introduce an enlarged frame field as in (1.2), but now with indices a, b, \dots taking $10 + n$ values and with the upper-left block of $\hat{\eta}_{AB}$ being

$$\eta_{ab} = \begin{pmatrix} \eta_{\underline{a}\underline{b}} & 0 \\ 0 & \delta_{\underline{\alpha}\underline{\beta}} \end{pmatrix}. \quad (3.3)$$

Here and in the following we split flat indices as

$$A = (a, \bar{a}) = (\underline{a}, \underline{\alpha}, \bar{a}), \quad \underline{a} = 0, \dots, 9, \quad \underline{\alpha} = 1, \dots, n. \quad (3.4)$$

The frame field is constrained by requiring that the tangent space metric \mathcal{G}_{AB} still satisfies (2.6), which reads explicitly

$$\mathcal{G}_{a\bar{b}} = 0, \quad \mathcal{G}_{\underline{a}\underline{b}} = \eta_{\underline{a}\underline{b}}, \quad \mathcal{G}_{\bar{a}\bar{b}} = -\eta_{\bar{a}\bar{b}}, \quad \mathcal{G}_{\underline{\alpha}\underline{\beta}} = \delta_{\underline{\alpha}\underline{\beta}}. \quad (3.5)$$

We can then choose a gauge and parametrize the frame field as follows

$$E_A^M = \begin{pmatrix} E_{\underline{a}i} & E_{\underline{\alpha}}^\beta & E_{\bar{a}}^i \\ E_{\underline{\alpha}i} & E_{\underline{\alpha}}^\beta & E_{\bar{a}}^i \\ E_{\bar{a}i} & E_{\bar{a}}^\beta & E_{\bar{a}}^i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_{i\underline{a}} - e_{\underline{a}}^k c_{ki} & -e_{\underline{a}}^k A_k^\beta & e_{\bar{a}}^i \\ \sqrt{2} A_{i\underline{\alpha}} & \sqrt{2} \delta_{\underline{\alpha}}^\beta & 0 \\ -e_{i\bar{a}} - e_{\bar{a}}^k c_{ki} & -e_{\bar{a}}^k A_k^\beta & e_{\bar{a}}^i \end{pmatrix}, \quad (3.6)$$

where we defined $c_{ij} = b_{ij} + \frac{1}{2} A_i^\alpha A_{j\alpha}$, and we freely raise and lower gauge group indices with the Kronecker delta $\delta_{\alpha\beta}$.

All results of the frame formalism reviewed in sec. 2.1 extend directly to the present generalization. In particular, all statements about determined connection components can be readily applied. Moreover, the supersymmetric extension (2.27) is well-defined for these extended fields in that the gamma matrices $\gamma^{\bar{a}}$ and all spinor indices are still to be interpreted with respect to $O(1, 9)$. The check of supersymmetric invariance and closure of the supersymmetry transformations immediately generalizes to the present case, as it is never used whether a takes 10 or $10 + n$ values. Assuming the parametrization (3.6) and setting $\tilde{\partial}^i = \partial_\alpha = 0$ we compute the following connection components:

$$\begin{aligned} \omega_{\underline{a}\bar{b}\bar{c}} &= -(\omega_{\underline{a}\bar{b}\bar{c}}^L(e) - \frac{1}{2} \hat{H}_{\underline{a}\bar{b}\bar{c}}), & \omega_{\bar{a}\underline{b}\underline{c}} &= \omega_{\bar{a}\underline{b}\underline{c}}^L(e) + \frac{1}{2} \hat{H}_{\bar{a}\underline{b}\underline{c}}, \\ \omega_{[\bar{a}\bar{b}\bar{c}]} &= -(\omega_{[\bar{a}\bar{b}\bar{c}]}^L(e) - \frac{1}{6} \hat{H}_{\bar{a}\bar{b}\bar{c}}), \\ \omega_{\bar{b}\bar{c}}^\alpha &= \frac{1}{\sqrt{2}} F_{\bar{b}\bar{c}}^\alpha, & \omega_{\bar{b}\underline{a}}^\alpha &= -\omega_{\bar{b}}^\alpha{}_{\underline{a}} = \frac{1}{\sqrt{2}} F_{\bar{b}\underline{a}}^\alpha, \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} F_{ab}^\alpha &= e_a^i e_b^j (\partial_i A_j^\alpha - \partial_j A_i^\alpha), \\ \hat{H}_{abc} &= 3e_a^i e_b^j e_c^k (\partial_{[i} b_{jk]} - A_{[i}^\alpha \partial_j A_{k]\alpha}). \end{aligned} \quad (3.8)$$

Thus, we obtained the abelian field strength of the gauge fields A_i^α and the required Chern-Simons modification of the field strength H .

3.2 Reduction to $\mathcal{N} = 1$ Supergravity with n vector multiplets

We will now show that the $\mathcal{N} = 1$ double field theory action with tangent space symmetry $O(1, 9 + n) \times O(1, 9)$ reproduces standard $\mathcal{N} = 1$ supergravity with n abelian vector multiplets upon setting $\tilde{\partial}^i = \partial_\alpha = 0$. Let us first recall $\mathcal{N} = 1$ supergravity coupled to n vector multiplets

$$(A_i^\alpha, \chi^\alpha), \quad \alpha = 1, \dots, n. \quad (3.9)$$

The action is given by

$$\begin{aligned} S = \int d^{10}x e e^{-2\phi} & \left[\left(R + 4\partial^i \phi \partial_i \phi - \frac{1}{12} \hat{H}^{ijk} \hat{H}_{ijk} - \frac{1}{4} F_{\alpha ij} F^{\alpha ij} \right) \right. \\ & - \bar{\psi}_i \gamma^{ijk} D_j \psi_k - 2\bar{\lambda} \gamma^i D_i \lambda - \frac{1}{2} \bar{\chi}^\alpha \not{D} \chi_\alpha \\ & + 2\bar{\psi}^i (\partial_i \phi) \gamma^j \psi_j - \bar{\psi}_i (\not{\partial} \phi) \gamma^i \lambda - \frac{1}{4} \bar{\chi}_\alpha \gamma^i \gamma^{jk} F_{jk}{}^\alpha (\psi_i + \frac{1}{6} \gamma_i \lambda) \\ & \left. + \frac{1}{24} \hat{H}_{ijk} \left(\bar{\psi}_m \gamma^{mijkn} \psi_n + 6\bar{\psi}^i \gamma^j \psi^k - 2\bar{\psi}_m \gamma^{ijk} \gamma^m \lambda + \frac{1}{2} \bar{\chi}^\alpha \gamma^{ijk} \chi_\alpha \right) \right], \end{aligned} \quad (3.10)$$

where \hat{H}_{ijk} is the H -field strength modified by the Chern-Simons 3-form, as in (3.8). This action is invariant under the supersymmetry transformations:

$$\begin{aligned} \delta_\epsilon e_i{}^a &= \frac{1}{2} \bar{\epsilon} \gamma^a \psi_i - \frac{1}{4} \bar{\epsilon} \lambda e_i{}^a, \\ \delta_\epsilon \phi &= -\bar{\epsilon} \lambda, \quad \delta_\epsilon A_i{}^\alpha = \frac{1}{2} \bar{\epsilon} \gamma_i \chi^\alpha, \quad \delta_\epsilon \chi^\alpha = -\frac{1}{4} \gamma^{ij} F_{ij}{}^\alpha \epsilon \\ \delta_\epsilon \psi_i &= D_i \epsilon - \frac{1}{8} \gamma_i (\not{\partial} \phi) \epsilon + \frac{1}{96} (\gamma_i{}^{klm} - 9\delta_i{}^k \gamma^{lm}) \hat{H}_{klm} \epsilon, \\ \delta_\epsilon \lambda &= -\frac{1}{4} (\not{\partial} \phi) \epsilon + \frac{1}{48} \gamma^{ijk} \hat{H}_{ijk} \epsilon, \\ \delta_\epsilon b_{ij} &= \frac{1}{2} (\bar{\epsilon} \gamma_i \psi_j - \bar{\epsilon} \gamma_j \psi_i) - \frac{1}{2} \bar{\epsilon} \gamma_{ij} \lambda + \frac{1}{2} \bar{\epsilon} \gamma_{[i} \chi^\alpha A_{j]\alpha}. \end{aligned} \quad (3.11)$$

Next, we perform the same field redefinition (2.56) as for the minimal theory. We obtain for the action

$$\begin{aligned} S_F = \int d^{10}x e e^{-2d} & \left[-\bar{\Psi}^j \gamma^i D_i \Psi_j + 2\bar{\Psi}^i D_i \rho + \bar{\rho} \gamma^i D_i \rho - \frac{1}{2} \bar{\chi}^\alpha \gamma^i D_i \chi_\alpha - \frac{1}{4} \bar{\chi}^\alpha \gamma^{jk} F_{jk\alpha} \rho \right. \\ & \left. - \bar{\chi}^\alpha \gamma^k F_{ik\alpha} \Psi^i + \frac{1}{4} \bar{\Psi}^i \hat{\not{H}} \Psi_i - \frac{1}{4} \bar{\rho} \hat{\not{H}} \rho + \frac{1}{2} \hat{H}_{ijk} \bar{\Psi}^i \gamma^j \Psi^k + \frac{1}{4} \hat{H}_{ijk} \bar{\rho} \gamma^{ij} \Psi^k + \frac{1}{8} \bar{\chi}^\alpha \hat{\not{H}} \chi_\alpha \right], \end{aligned} \quad (3.12)$$

and the supersymmetry transformations are given by

$$\begin{aligned} \delta_\epsilon e_i{}^a &= \frac{1}{2} \bar{\epsilon} \gamma^a \Psi_i, \quad \delta_\epsilon \Psi_i = D_i(\hat{\omega}) \epsilon, \\ \delta_\epsilon b_{ij} &= \bar{\epsilon} \gamma_{[i} \Psi_{j]} + \frac{1}{2} \bar{\epsilon} \gamma_{[i} \chi A_{j]}, \\ \delta_\epsilon d &= -\frac{1}{4} \bar{\epsilon} \rho, \quad \delta_\epsilon \rho = \gamma^i D_i \epsilon - \frac{1}{24} \hat{H}_{ijk} \gamma^{ijk} \epsilon - (\not{\partial} \phi) \epsilon, \\ \delta_\epsilon A_i{}^\alpha &= \frac{1}{2} \bar{\epsilon} \gamma_i \chi^\alpha, \quad \delta_\epsilon \chi^\alpha = -\frac{1}{4} \gamma^{ij} F_{ij}{}^\alpha \epsilon. \end{aligned} \quad (3.13)$$

Let us now verify that the above action and supersymmetry rules are reproduced by supersymmetric double field theory for $\tilde{\partial}^i = \partial_\alpha = 0$. Here, our discussion will be a little briefer than above because it suffices to focus on the new structures involving the gauge vectors and gauginos. It turns out that the comparison requires the identification

$$\Psi_a = (\Psi_{\underline{a}}, \Psi_{\underline{\alpha}}) \equiv (e_{\underline{a}}^i \Psi_i, \frac{1}{\sqrt{2}} \chi_{\underline{\alpha}}), \quad (3.14)$$

i.e., the gauginos are naturally identified with the additional components of the ‘gravitino’.

We start with the supersymmetry transformations. The gaugino variation $\delta_\epsilon \chi^\alpha$ can be obtained by considering

$$\delta_\epsilon \Psi_{\underline{\alpha}} = \frac{1}{\sqrt{2}} \delta_\epsilon \chi_{\underline{\alpha}} = \nabla_{\underline{\alpha}} \epsilon = \left(\sqrt{2} E_{\underline{\alpha}}^i \partial_i \epsilon - \frac{1}{4} \omega_{\underline{\alpha} \bar{b} \bar{c}} \gamma^{\bar{b} \bar{c}} \epsilon \right) = -\frac{1}{4\sqrt{2}} F_{\bar{b} \bar{c} \underline{\alpha}} \gamma^{\bar{b} \bar{c}} \epsilon, \quad (3.15)$$

where we used (3.7) and $E_{\underline{\alpha}}^i = 0$ for the gauge choice (3.6). We read off

$$\delta_\epsilon \chi^\alpha = -\frac{1}{4} F_{\bar{b} \bar{c}}^\alpha \gamma^{\bar{b} \bar{c}} \epsilon. \quad (3.16)$$

Comparison with (3.13) shows that we obtained the expected supersymmetry variation. For the supersymmetry variations of the vielbein e_i^a , the b -field and the gauge vectors we compute as in (2.64) the variation of the gauge-fixed frame field (3.6)

$$\Delta_\epsilon E_{\underline{a} \bar{b}} = e_{\bar{b}}^i \delta_\epsilon e_{i \underline{a}} + e_{\underline{a}}^i \delta_\epsilon e_{i \bar{b}} - \frac{1}{2} e_{\underline{a}}^i e_{\bar{b}}^j \delta_\epsilon b_{ij} - \frac{1}{2} e_{\underline{a}}^i e_{\bar{b}}^j A_{[i}{}^\alpha \delta_\epsilon A_{j] \underline{\alpha}} = -\frac{1}{2} \bar{\epsilon} \gamma_{\bar{b}} \Psi_{\underline{a}}, \quad (3.17)$$

and

$$\Delta_\epsilon E_{\underline{\alpha} \bar{b}} = \frac{\sqrt{2}}{2} e_{\bar{b}}^i \delta_\epsilon A_{i \underline{\alpha}} = -\frac{1}{2\sqrt{2}} \bar{\epsilon} \gamma_{\bar{b}} \chi_{\underline{\alpha}}. \quad (3.18)$$

Combining these two gives the required supersymmetry transformations (3.13).

Let us now turn to the action and show that it produces the required χ -dependent terms. For the first fermionic term in (1.4) we obtain

$$\begin{aligned} -\bar{\Psi}^a \gamma^{\bar{b}} \nabla_{\bar{b}} \Psi_a \Big|_\chi &= -\frac{1}{2} \bar{\chi}^\alpha \gamma^{\bar{b}} D_{\bar{b}} \chi_{\underline{\alpha}} + \frac{1}{8} \bar{\chi}^\alpha \not{H} \chi_{\underline{\alpha}} - \bar{\Psi}^{\underline{a}} \gamma^{\bar{b}} \omega_{\bar{b} \underline{a}}{}^\alpha \Psi_{\underline{\alpha}} - \bar{\Psi}^{\underline{\alpha}} \gamma^{\bar{b}} \omega_{\bar{b} \underline{\alpha}}{}^a \Psi_a \\ &= -\frac{1}{2} \bar{\chi}^\alpha \gamma^{\bar{b}} D_{\bar{b}} \chi_{\underline{\alpha}} + \frac{1}{8} \bar{\chi}^\alpha \not{H} \chi_{\underline{\alpha}} - \bar{\chi}_{\underline{\alpha}} \gamma^{\bar{b}} F_{\bar{b} \underline{a}}{}^\alpha \Psi^{\underline{a}}, \end{aligned} \quad (3.19)$$

where we used in the first line that the last two terms are equal. The second fermionic term in the action (1.4) does not give any χ -dependent contribution. The third term reads

$$2\bar{\Psi}^a \nabla_a \rho \Big|_\chi = 2\bar{\Psi}^{\underline{a}} \nabla_{\underline{a}} \rho = \frac{2}{\sqrt{2}} \bar{\chi}^\alpha \left(-\frac{1}{4} \omega_{\underline{\alpha} \bar{b} \bar{c}} \gamma^{\bar{b} \bar{c}} \right) \rho = -\frac{1}{4} \bar{\chi}_{\underline{\alpha}} \gamma^{\bar{b} \bar{c}} F_{\bar{b} \bar{c}}^\alpha \rho, \quad (3.20)$$

reproducing the required coupling in (3.12). Thus, we have shown that all new χ - and F -dependent terms due to the coupling of vector multiplets are precisely reproduced by the extended connections of the $O(1, 9+n) \times O(1, 9)$ tangent space symmetry.

4 Conclusions

In this paper we have constructed the $\mathcal{N} = 1$ supersymmetric extension of double field theory for $D = 10$. This theory features two copies of the local Lorentz group as tangent space symmetries, under which the fermions naturally transform. Interestingly, the generalization to the coupling of n abelian vector multiplets amounts only to the extension of the T-duality group to $O(10 + n, 10)$ and, correspondingly, to the extension of the tangent space group to $O(1, 9 + n) \times O(1, 9)$. The ‘gravitino’ Ψ_a thereby receives n additional components that can be identified with the gauginos. Apart from exhibiting a further ‘unification’ of the massless sector of heterotic superstring theory, this formulation provides a significant technical simplification of the effective action, as should be apparent by comparing (3.10) with (1.4). Moreover, the proof of supersymmetric invariance (up to the higher order fermi terms) is much simpler than in the standard formulation, being essentially reduced to a two-line calculation in (2.33).

On a technical level it is interesting to note that the connections emerging naturally in double field theory, $\omega^\pm = \omega^L \pm \frac{1}{2}H$, have appeared in different contexts in string theory. For instance, they turn out to be very useful for constructing supersymmetric higher-derivative invariants [28], and it would be interesting to understand the significance of this relation.

This work can be extended into many directions. First, the generalization to non-abelian vector multiplets is necessary in order to describe the full massless sector of the heterotic superstring. For the bosonic sector we described in [9] also the non-abelian generalization, but the formalism and physical interpretation is different. We hope to come back to this problem.

Next, one should construct the $\mathcal{N} = 2$ supersymmetric extension of the type II double field theory constructed in [10]. The recent work [25] completes the corresponding construction in generalized geometry, but there are a few subtleties in double field theory that we hope to address and resolve in the near future.

Finally, the recent results [13] on similar constructions for M-theory or 11-dimensional supergravity suggest that an analogous supersymmetric extension is possible there. Here we note that in [12] the supersymmetry variations of 11-dimensional supergravity (in a certain truncation to $D = 7$) have already been written in an $E_{7(7)}$ and $SU(8)$ covariant way, and it would be nice to show that the corresponding action is supersymmetric modulo a covariant constraint.

Acknowledgments

We would like to thank Barton Zwiebach for many helpful discussions.

This work is supported by the U.S. Department of Energy (DoE) under the cooperative research agreement DE-FG02-05ER41360, the DFG Transregional Collaborative Research Centre TRR 33 and the DFG cluster of excellence “Origin and Structure of the Universe”. The work of SK is supported in part by a Samsung Scholarship.

A Identities for the curvature tensors

In this appendix we present some details of the derivation of the identities (2.34). We start with the second one, involving the Ricci tensor, and compute

$$\begin{aligned}
[\gamma^{\bar{a}}\nabla_{\bar{a}}, \nabla_b]\epsilon &= \gamma^{\bar{a}}\nabla_{\bar{a}}\nabla_b\epsilon - \nabla_b(\gamma^{\bar{a}}\nabla_{\bar{a}}\epsilon) \\
&= \left(\gamma^{\bar{a}}E_{\bar{a}} - \frac{1}{4}\omega_{\bar{a}\bar{e}\bar{f}}\gamma^{\bar{a}\bar{e}\bar{f}} - \frac{1}{2}\omega_{\bar{a}\bar{e}}^{\bar{a}}\gamma^{\bar{e}}\right)\left(E_b - \frac{1}{4}\omega_{b\bar{c}\bar{d}}\gamma^{\bar{c}\bar{d}}\right)\epsilon + \gamma^{\bar{a}}\omega_{\bar{a}b}^f\left(E_f - \frac{1}{4}\omega_{f\bar{c}\bar{d}}\gamma^{\bar{c}\bar{d}}\right)\epsilon \\
&\quad - \left(E_b - \frac{1}{4}\omega_{b\bar{c}\bar{d}}\gamma^{\bar{c}\bar{d}}\right)\left(\gamma^{\bar{a}}E_{\bar{a}} - \frac{1}{4}\omega_{\bar{a}\bar{e}\bar{f}}\gamma^{\bar{a}\bar{e}\bar{f}} - \frac{1}{2}\omega_{\bar{a}\bar{e}}^{\bar{a}}\gamma^{\bar{e}}\right)\epsilon.
\end{aligned} \tag{A.1}$$

Our strategy is now to work out the various powers $\gamma^{(p)}$ of gamma matrices separately and to show that all except $\gamma^{(1)}$ cancel. The non-vanishing contribution will then be shown to be related to the Ricci tensor. To this end we use the following identities for the product of (antisymmetrized) gamma matrices

$$\gamma^{\bar{a}}\gamma_{\bar{b}} = \gamma^{\bar{a}}_{\bar{b}} - \delta^{\bar{a}}_{\bar{b}}, \tag{A.2}$$

$$\gamma^{\bar{a}\bar{b}}\gamma_{\bar{c}} = \gamma^{\bar{a}\bar{b}}_{\bar{c}} + 2\delta^{[\bar{a}}_{\bar{c}}\gamma^{\bar{b}]}, \tag{A.3}$$

$$\gamma^{\bar{a}\bar{b}\bar{c}}\gamma_{\bar{d}} = \gamma^{\bar{a}\bar{b}\bar{c}}_{\bar{d}} - 3\delta^{[\bar{a}}_{\bar{d}}\gamma^{\bar{b}\bar{c}]}, \tag{A.4}$$

$$\gamma^{\bar{a}\bar{b}}\gamma_{\bar{c}\bar{d}} = \gamma^{\bar{a}\bar{b}}_{\bar{c}\bar{d}} + 4\delta^{[\bar{a}}_{\bar{c}}\gamma^{\bar{b}]}_{\bar{d}} - 2\delta^{[\bar{a}}_{\bar{c}}\delta^{\bar{b}]}_{\bar{d}}, \tag{A.5}$$

$$\gamma^{\bar{a}\bar{b}\bar{c}}\gamma_{\bar{d}\bar{e}} = \gamma^{\bar{a}\bar{b}\bar{c}}_{\bar{d}\bar{e}} - 6\delta^{[\bar{a}}_{\bar{d}}\gamma^{\bar{b}\bar{c}]}_{\bar{e}} - 6\delta^{[\bar{a}}_{\bar{d}}\delta^{\bar{b}]}_{\bar{e}}\gamma^{\bar{c]}, \tag{A.6}$$

$$\gamma^{\bar{a}\bar{b}\bar{c}}\gamma_{\bar{d}\bar{e}\bar{f}} = \gamma^{\bar{a}\bar{b}\bar{c}}_{\bar{d}\bar{e}\bar{f}} - 9\delta^{[\bar{a}}_{\bar{d}}\gamma^{\bar{b}\bar{c}]}_{\bar{e}\bar{f}} - 18\delta^{[\bar{a}}_{\bar{d}}\delta^{\bar{b}]}_{\bar{e}}\gamma^{\bar{c}]}_{\bar{f}} + 6\delta^{[\bar{a}}_{\bar{d}}\delta^{\bar{b}]}_{\bar{e}}\delta^{\bar{c}]}_{\bar{f}}, \tag{A.7}$$

where we recall that indices are raised and lowered with $\mathcal{G}_{\bar{a}\bar{b}} = -\eta_{\bar{a}\bar{b}}$.

Let us now start the computation. First, the $\gamma^{(5)}$ terms cancel:

$$\frac{1}{16}\omega_{\bar{a}\bar{e}\bar{f}}\omega_{b\bar{c}\bar{d}}\gamma^{\bar{a}\bar{e}\bar{f}\bar{c}\bar{d}} - \frac{1}{16}\omega_{\bar{a}\bar{e}\bar{f}}\omega_{b\bar{c}\bar{d}}\gamma^{\bar{c}\bar{d}\bar{a}\bar{e}\bar{f}} = 0. \tag{A.8}$$

Second, it is easy to see by inspection that there are no $\gamma^{(4)}$ terms. Next, collecting terms with $\gamma^{(3)}$ we find

$$\begin{aligned}
&\left[-\frac{1}{4}E_{\bar{a}}\omega_{b\bar{c}\bar{d}} + \frac{1}{4}E_b\omega_{[\bar{a}\bar{c}\bar{d}]} + \frac{3}{4}\omega_{[\bar{e}\bar{c}\bar{d}]} \omega_{b\bar{a}}^{\bar{e}} - \frac{1}{4}\omega_{e\bar{c}\bar{d}}\omega_{\bar{a}b}^e\right]\gamma^{\bar{a}\bar{c}\bar{d}}\epsilon \\
&= \frac{1}{4}\left[E_{\bar{a}}\Omega_{b\bar{c}\bar{d}} - E_b\Omega_{[\bar{a}\bar{b}\bar{d}]} - \Omega_{[\bar{e}\bar{c}\bar{d}]} \Omega_{\bar{a}b}^{\bar{e}} - \Omega_{e\bar{c}\bar{d}}\Omega_{\bar{a}b}^e\right]\gamma^{\bar{a}\bar{c}\bar{d}}\epsilon,
\end{aligned} \tag{A.9}$$

where we inserted in the second line the solutions for the connections. Inserting now the explicit expressions for Ω it is a straightforward though somewhat lengthy calculation to verify that this vanishes. It is again easy to see that there are no $\gamma^{(2)}$ terms. So we finally have to work out the terms proportional to $\gamma^{(1)}$, for which we find

$$\begin{aligned}
&\gamma^{\bar{a}}\left[(E_{\bar{a}}E_b^M)E_M^C - (E_bE_{\bar{a}}^M)E_M^C\right]E_C + \omega_{\bar{a}b}^cE_c - \omega_{b\bar{a}}^{\bar{c}}E_{\bar{c}}\epsilon \\
&- \frac{1}{2}\gamma^{\bar{a}}\left[E_{\bar{c}}\omega_{b\bar{a}}^{\bar{c}} - E_b\omega_{\bar{c}\bar{a}}^{\bar{c}} + \omega_{d\bar{a}}^{\bar{c}}\omega_{\bar{c}b}^d - \omega_{b\bar{a}}^{\bar{d}}\omega_{\bar{c}d}^{\bar{c}}\right]\epsilon.
\end{aligned} \tag{A.10}$$

The terms in the first line vanish as a consequence of the torsion constraint (2.11): Using $f_{ABC} \equiv (E_A E_B^M) E_{CM}$, the torsion constraint reads

$$(f_{\bar{a}b}^C - f_{b\bar{a}}^C) E_C + \omega_{\bar{a}b}^c E_c - \omega_{b\bar{a}}^{\bar{c}} E_{\bar{c}} = (\Omega_{\bar{a}b}^C + 2\omega_{[\bar{a}b]}^C) E_C = 0, \quad (\text{A.11})$$

where we used the strong constraint (2.4). Thus the final result is

$$\begin{aligned} [\gamma^{\bar{a}} \nabla_{\bar{a}} \gamma^{\bar{b}} \nabla_{\bar{b}} - \nabla^a \nabla_a] \epsilon &= -\frac{1}{2} \gamma^{\bar{a}} \left[E_{\bar{c}} \omega_{b\bar{a}}^{\bar{c}} - E_b \omega_{\bar{c}\bar{a}}^{\bar{c}} + \omega_{d\bar{a}}^{\bar{c}} \omega_{\bar{c}b}^d - \omega_{b\bar{a}}^{\bar{d}} \omega_{\bar{c}d}^{\bar{c}} \right] \epsilon \\ &= -\frac{1}{2} \gamma^{\bar{a}} \mathcal{R}_{b\bar{a}} \epsilon, \end{aligned} \quad (\text{A.12})$$

as claimed in (2.34).

Let us now turn to the second identity in (2.34) involving the scalar curvature. We compute

$$\begin{aligned} (\gamma^{\bar{a}} \nabla_{\bar{a}} \gamma^{\bar{b}} \nabla_{\bar{b}} - \nabla^a \nabla_a) \epsilon &= \left(\gamma^{\bar{a}} E_{\bar{a}} - \frac{1}{4} \omega_{\bar{a}\bar{e}\bar{f}} \gamma^{\bar{a}\bar{e}\bar{f}} - \frac{1}{2} \omega_{\bar{a}\bar{e}}^{\bar{a}} \gamma^{\bar{e}} \right) \left(\gamma^{\bar{b}} E_{\bar{b}} - \frac{1}{4} \omega_{\bar{b}\bar{c}\bar{d}} \gamma^{\bar{b}\bar{c}\bar{d}} - \frac{1}{2} \omega_{\bar{b}\bar{c}}^{\bar{b}} \gamma^{\bar{c}} \right) \epsilon \\ &\quad - \left(E^a - \frac{1}{4} \omega_{\bar{e}\bar{f}}^a \gamma^{\bar{e}\bar{f}} \right) \left(E_a - \frac{1}{4} \omega_{a\bar{c}\bar{d}} \gamma^{\bar{c}\bar{d}} \right) \epsilon - \omega_a^{ab} \left(E_b - \frac{1}{4} \omega_{b\bar{c}\bar{d}} \gamma^{\bar{c}\bar{d}} \right) \epsilon. \end{aligned} \quad (\text{A.13})$$

As above, we work out the various powers $\gamma^{(p)}$ of gamma matrices separately, which here are non-trivial only for even p , and then show that only the scalar part survives. The $\gamma^{(6)}$ terms are easily seen to cancel,

$$\frac{1}{16} \omega_{\bar{a}\bar{e}\bar{f}} \omega_{\bar{b}\bar{c}\bar{d}} \gamma^{\bar{a}\bar{e}\bar{f}\bar{b}\bar{c}\bar{d}} = \frac{1}{16} \omega_{\bar{a}\bar{e}\bar{f}} \omega_{\bar{b}\bar{c}\bar{d}} \gamma^{\bar{b}\bar{c}\bar{d}\bar{a}\bar{e}\bar{f}} = -\frac{1}{16} \omega_{\bar{a}\bar{e}\bar{f}} \omega_{\bar{b}\bar{c}\bar{d}} \gamma^{\bar{a}\bar{e}\bar{f}\bar{b}\bar{c}\bar{d}} = 0. \quad (\text{A.14})$$

We have verified that the $\gamma^{(4)}$ and $\gamma^{(2)}$ structures cancel upon insertion of the explicit expressions for the determined connections, which is a rather lengthy computation that we do not display here. Let us finally turn to the scalar part (without gamma matrices). It reads

$$\left[-E^A E_A + \omega_a^{ba} E_b + \omega_{\bar{a}}^{\bar{b}\bar{a}} E_{\bar{b}} \right] \epsilon + \frac{1}{2} \left[E_{\bar{a}} \omega_{\bar{b}}^{\bar{a}\bar{b}} + \frac{3}{4} \omega_{[\bar{a}\bar{b}\bar{c}]} \omega^{[\bar{a}\bar{b}\bar{c}]} - \frac{1}{2} \omega_{\bar{a}}^{\bar{c}\bar{a}} \omega_{\bar{b}\bar{c}}^{\bar{b}} + \frac{1}{4} \omega_{a\bar{b}\bar{c}} \omega^{a\bar{b}\bar{c}} \right] \epsilon. \quad (\text{A.15})$$

The terms in the first square bracket vanish. To see this, we write it out and insert the determined connections,

$$\begin{aligned} &\left[-\sqrt{2} (E^A E_A^M) \partial_M - \sqrt{2} (\partial_M E_b^M) E^b + 2(E_b d) E^b - \sqrt{2} (\partial_M E_{\bar{b}}^M) E^{\bar{b}} + 2(E_{\bar{b}} d) E^{\bar{b}} \right] \epsilon \\ &= \left[-\sqrt{2} (E^A E_A^M) \partial_M - \sqrt{2} (\partial_M E_B^M) E^{BN} \partial_N + 2(E_B d) E^B \right] \epsilon = 0. \end{aligned} \quad (\text{A.16})$$

Here we used the strong constraint, which implies that $E_B d E^B \epsilon = 0$. Therefore, the only non-vanishing contribution is the second bracket in (A.15), which is proportional to the scalar curvature (2.22). We have thus shown

$$(\gamma^{\bar{a}} \nabla_{\bar{a}} \gamma^{\bar{b}} \nabla_{\bar{b}} - \nabla^a \nabla_a) \epsilon = -\frac{1}{4} \mathcal{R} \epsilon, \quad (\text{A.17})$$

as claimed in (2.34).

References

- [1] C. Hull, B. Zwiebach, “Double Field Theory,” JHEP **0909**, 099 (2009). [arXiv:0904.4664 [hep-th]].
- [2] C. Hull, B. Zwiebach, “The Gauge algebra of double field theory and Courant brackets,” JHEP **0909**, 090 (2009). [arXiv:0908.1792 [hep-th]].
- [3] O. Hohm, C. Hull and B. Zwiebach, “Background independent action for double field theory,” JHEP **1007** (2010) 016 [arXiv:1003.5027 [hep-th]].
- [4] O. Hohm, C. Hull and B. Zwiebach, “Generalized metric formulation of double field theory,” JHEP **1008** (2010) 008 [arXiv:1006.4823 [hep-th]].
- [5] W. Siegel, “Superspace duality in low-energy superstrings,” Phys. Rev. D **48**, 2826 (1993) [arXiv:hep-th/9305073], “Two vierbein formalism for string inspired axionic gravity,” Phys. Rev. D **47**, 5453 (1993) [arXiv:hep-th/9302036].
- [6] A. A. Tseytlin, “Duality Symmetric Formulation Of String World Sheet Dynamics,” Phys. Lett. B **242**, 163 (1990); “Duality Symmetric Closed String Theory And Interacting Chiral Scalars,” Nucl. Phys. B **350**, 395 (1991).
- [7] O. Hohm, S. K. Kwak, “Frame-like Geometry of Double Field Theory,” J. Phys. A **A44**, 085404 (2011). [arXiv:1011.4101 [hep-th]],
- [8] S. K. Kwak, “Invariances and Equations of Motion in Double Field Theory,” JHEP **1010** (2010) 047 [arXiv:1008.2746 [hep-th]],
O. Hohm, “T-duality versus Gauge Symmetry,” arXiv:1101.3484 [hep-th],
O. Hohm, “On factorizations in perturbative quantum gravity,” JHEP **1104**, 103 (2011). [arXiv:1103.0032 [hep-th]],
B. Zwiebach, “Double Field Theory, T-Duality, and Courant Brackets,” [arXiv:1109.1782 [hep-th]].
- [9] O. Hohm, S. K. Kwak, “Double Field Theory Formulation of Heterotic Strings,” JHEP **1106**, 096 (2011). [arXiv:1103.2136 [hep-th]].
- [10] O. Hohm, S. K. Kwak, B. Zwiebach, “Unification of Type II Strings and T-duality,” Phys. Rev. Lett. **107**, 171603 (2011), [arXiv:1106.5452 [hep-th]], “Double Field Theory of Type II Strings,” JHEP **1109**, 013 (2011), [arXiv:1107.0008 [hep-th]].
- [11] O. Hohm and S. K. Kwak, “Massive Type II in Double Field Theory,” JHEP **1111**, 086 (2011) [arXiv:1108.4937 [hep-th]].
- [12] C. Hillmann, “Generalized E(7(7)) coset dynamics and D=11 supergravity,” JHEP **0903**, 135 (2009). [arXiv:0901.1581 [hep-th]].
- [13] D. S. Berman, M. J. Perry, “Generalized Geometry and M theory,” JHEP **1106**, 074 (2011). [arXiv:1008.1763 [hep-th]],
D. S. Berman, H. Godazgar, M. J. Perry, “SO(5,5) duality in M-theory and generalized

- geometry,” Phys. Lett. **B700**, 65-67 (2011). [arXiv:1103.5733 [hep-th]],
D. S. Berman, E. T. Musaev, M. J. Perry, “Boundary Terms in Generalized Geometry and doubled field theory,” [arXiv:1110.3097 [hep-th]],
D. S. Berman, H. Godazgar, M. Godazgar, M. J. Perry, “The Local symmetries of M-theory and their formulation in generalised geometry,” [arXiv:1110.3930 [hep-th]],
D. S. Berman, H. Godazgar, M. J. Perry, P. West, “Duality Invariant Actions and Generalised Geometry,” [arXiv:1111.0459 [hep-th]].
- [14] P. West, “ E_{11} , generalised space-time and IIA string theory,” Phys. Lett. **B696**, 403-409 (2011). [arXiv:1009.2624 [hep-th]],
A. Rocen, P. West, “E11, generalised space-time and IIA string theory: the R-R sector,” [arXiv:1012.2744 [hep-th]].
- [15] I. Jeon, K. Lee, J. -H. Park, “Differential geometry with a projection: Application to double field theory,” JHEP **1104**, 014 (2011). [arXiv:1011.1324 [hep-th]].
- [16] I. Jeon, K. Lee, J. -H. Park, “Stringy differential geometry, beyond Riemann,” Phys. Rev. **D84**, 044022 (2011). [arXiv:1105.6294 [hep-th]].
- [17] I. Jeon, K. Lee and J. -H. Park, “Double field formulation of Yang-Mills theory,” Phys. Lett. B **701** (2011) 260 [arXiv:1102.0419 [hep-th]].
- [18] I. Jeon, K. Lee, J. -H. Park, “Incorporation of fermions into double field theory,” JHEP **1111**, 025 (2011). [arXiv:1109.2035 [hep-th]].
- [19] M. B. Schulz, “T-folds, doubled geometry, and the SU(2) WZW model,” [arXiv:1106.6291 [hep-th]].
- [20] N. B. Copland, “Connecting T-duality invariant theories,” Nucl. Phys. **B854**, 575-591 (2012). [arXiv:1106.1888 [hep-th]], “A Double Sigma Model for Double Field Theory,” [arXiv:1111.1828 [hep-th]].
- [21] D. C. Thompson, “Duality Invariance: From M-theory to Double Field Theory,” JHEP **1108**, 125 (2011). [arXiv:1106.4036 [hep-th]].
- [22] C. Albertsson, S. -H. Dai, P. -W. Kao, F. -L. Lin, “Double Field Theory for Double D-branes,” JHEP **1109**, 025 (2011). [arXiv:1107.0876 [hep-th]].
- [23] D. Andriot, M. Larfors, D. Lust, P. Patalong, “A ten-dimensional action for non-geometric fluxes,” JHEP **1109**, 134 (2011). [arXiv:1106.4015 [hep-th]],
G. Aldazabal, W. Baron, D. Marques, C. Nunez, “The effective action of Double Field Theory,” JHEP **1111**, 052 (2011). [arXiv:1109.0290 [hep-th]],
D. Geissbuhler, “Double Field Theory and N=4 Gauged Supergravity,” [arXiv:1109.4280 [hep-th]].
- [24] B. de Wit, H. Nicolai, “Hidden Symmetry in $d = 11$ Supergravity,” Phys. Lett. **B155**, 47 (1985), “ $d = 11$ Supergravity with local SU(8) invariance,” Nucl. Phys. **B274**, 363 (1986).
- [25] A. Coimbra, C. Strickland-Constable, D. Waldram, “Supergravity as Generalised Geometry I: Type II Theories,” [arXiv:1107.1733 [hep-th]].

- [26] I. Jeon, K. Lee and J. -H. Park, “Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity,” arXiv:1112.0069 [hep-th].
- [27] E. Bergshoeff, M. de Roo, “Supersymmetric Chern-simons Terms In Ten-dimensions,” Phys. Lett. **B218** (1989) 210.
- [28] E. A. Bergshoeff and M. de Roo, ”The quartic effective action of the heterotic string and supersymmetry”, Nuclear Physics B 328 (1989) 439